

Renormalized asymptotic solutions of the Burgers equation and the Korteweg–de Vries equation

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Abstract. The Cauchy problem for the Burgers equation and the Korteweg–de Vries equation is considered. Uniform renormalized asymptotic solutions are constructed in cases of a large initial gradient and a perturbed initial weak discontinuity.

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1 Problem with a large initial gradient

A simplest model of the motion of continuum, which takes into account nonlinear effects and dissipation, is the equation of nonlinear diffusion

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad \varepsilon > 0, \quad (1.1)$$

for the first time presented by J. Burgers [1]. This equation is used in studying the evolution of a wide class of physical systems and probabilistic processes, for example, acoustic waves in fluid and gas [2].

Let us consider the Cauchy problem

$$u(x, 0, \varepsilon, \rho) = \Lambda(x\rho^{-1}), \quad t = 0, \quad x \in \mathbb{R}, \quad \rho > 0, \quad (1.2)$$

where ρ is another small parameter. The initial function Λ is smooth and it has finite limits

$$\Lambda_0^\pm = \lim_{s \rightarrow \pm\infty} \Lambda(s).$$

We see that the origin ($x = 0$ and $t = 0$) must be a singular point in the problem under consideration. It is known that the behavior of solutions of differential equations with a small parameter at a higher derivative in a neighborhood of a singular point sometimes becomes self-similar. Then it is effective to analyze the solution using the renormalization group method [3]. The success of applying renormalization is confirmed by examples from hydrodynamics [4], mechanics [5], and asymptotic analysis of solutions of nonlinear parabolic equations [6].

In paper [7], for the solution of a more general quasi-linear parabolic equation

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0,$$

with condition (1.2) the following asymptotic formula is obtained:

$$u(x, t, \varepsilon, \rho) = \frac{1}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} \Gamma \left(\frac{x - \rho s}{\varepsilon}, \frac{t}{\varepsilon} \right) \Lambda'(s) ds + O \left(\left(\frac{\rho}{\varepsilon} \right)^{1/4} \right) \quad (1.3)$$

as $\varepsilon \rightarrow 0$ and $\rho/\varepsilon \rightarrow 0$ uniformly in the strip $\{(x, t) : x \in \mathbb{R}, 0 \leq t \leq T\}$, where Γ is the solution of the limit problem in the inner variables $\eta = x/\varepsilon, \theta = t/\varepsilon$

$$\frac{\partial \Gamma}{\partial \theta} + \frac{\partial \varphi(\Gamma)}{\partial \eta} - \frac{\partial^2 \Gamma}{\partial \eta^2} = 0, \quad \Gamma(\eta, 0) = \begin{cases} \Lambda_0^-, & \eta < 0, \\ \Lambda_0^+, & \eta > 0. \end{cases}$$

The procedure of obtaining expression (1.3) can be also seen below in details on the example of the KdV equation. The difference consists in the scales of the inner variables. For the solution of the Burgers equation (1.1) as $\varepsilon \rightarrow 0$ and $\rho/\varepsilon \rightarrow 0$ in the strip $\{(x, t) : x \in \mathbb{R}, 0 \leq t \leq T\}$ there holds the asymptotic formula

$$\begin{aligned} u(x, t, \varepsilon, \rho) = & \int_{-\infty}^{\infty} \frac{\Lambda'(s)}{\Lambda_0^+ - \Lambda_0^-} \left[\Lambda_0^+ \exp \left(\frac{t(\Lambda_0^+)^2 - 2\Lambda_0^+ x}{4\varepsilon} + \frac{\Lambda_0^+ \rho s}{2\varepsilon} \right) \operatorname{erfc} \left(\frac{\Lambda_0^+ t - x + \rho s}{2\sqrt{\varepsilon t}} \right) + \right. \\ & + \Lambda_0^- \exp \left(\frac{t(\Lambda_0^-)^2 - 2\Lambda_0^- x}{4\varepsilon} + \frac{\Lambda_0^- \rho s}{2\varepsilon} \right) \operatorname{erfc} \left(\frac{x - \Lambda_0^- t - \rho s}{2\sqrt{\varepsilon t}} \right) \Big] \times \\ & \times \left[\exp \left(\frac{t(\Lambda_0^+)^2 - 2\Lambda_0^+ x}{4\varepsilon} + \frac{\Lambda_0^+ \rho s}{2\varepsilon} \right) \operatorname{erfc} \left(\frac{\Lambda_0^+ t - x + \rho s}{2\sqrt{\varepsilon t}} \right) + \right. \\ & \left. + \exp \left(\frac{t(\Lambda_0^-)^2 - 2\Lambda_0^- x}{4\varepsilon} + \frac{\Lambda_0^- \rho s}{2\varepsilon} \right) \operatorname{erfc} \left(\frac{x - \Lambda_0^- t - \rho s}{2\sqrt{\varepsilon t}} \right) \right]^{-1} ds + O(\mu^{1/4}). \end{aligned}$$

2 Renormalized weak discontinuity

In the present paper, we give an asymptotic solution of the Burgers equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \quad (2.1)$$

with the initial condition

$$u(x, 0) = \varepsilon \Lambda \left(\frac{x}{\varepsilon} \right), \quad t = 0, \quad x \in \mathbb{R}, \quad (2.2)$$

where $\Lambda(s) \rightarrow -s$ as $s \rightarrow -\infty$ and $\Lambda(s) \rightarrow 0$ as $s \rightarrow +\infty$.

By analogy with (1.3) we postulate the expression

$$R(x, t, \varepsilon) = \int_{-\infty}^{\infty} \Lambda''(s) u_0(x - \varepsilon s, t) ds,$$

where $u_0 = -2\Psi_x/\Psi$ is the solution of the limit problem with a weak discontinuity [8],

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} = \frac{\partial^2 u_0}{\partial x^2}, \quad u_0(x, 0) = \begin{cases} -x, & x < 0, \\ 0, & x \geq 0. \end{cases}$$

Ψ is the solution of the heat equation $\Psi_t = \Psi_{xx}$. Thus, we obtain a renormalized asymptotic solution in the form

$$R(x, t, \varepsilon) = -\frac{2}{\varepsilon} \int_{-\infty}^{\infty} \Lambda'''(s) \ln \Psi(x - \varepsilon s, t) ds.$$

Theorem 1. *Let $\Lambda(s)$ be a thrice differentiable function and let*

$$\frac{d\Lambda(s)}{ds} = -1 + O(|s|^{-3}), \quad s \rightarrow -\infty, \quad (2.3)$$

$$\frac{d\Lambda(s)}{ds} = O(s^{-3}), \quad s \rightarrow +\infty, \quad (2.4)$$

$$\frac{d^2\Lambda(s)}{ds^2} = O(s^{-4}), \quad s \rightarrow \infty. \quad (2.5)$$

Then the function

$$R(x, t, \varepsilon) = -\frac{2}{\varepsilon} \int_{-\infty}^{\infty} \frac{d^3\Lambda(s)}{ds^3} \ln \Psi(x - \varepsilon s, t) ds$$

is an asymptotic solution of the Burgers equation (2.1), where

$$\Psi(x, t) = \int_{-\infty}^0 \exp\left(-\frac{(x-\sigma)^2}{4t} + \frac{\sigma^2}{4}\right) d\sigma + \int_0^{\infty} \exp\left(-\frac{(x-\sigma)^2}{4t}\right) d\sigma.$$

Proof. Substituting $u = R(x, t, \varepsilon)$ into (2.1), we find

$$\begin{aligned} R_t + RR_x - R_{xx} &= \frac{4}{\varepsilon^2} \int_{-\infty}^{\infty} \Lambda'''(s) \frac{\Psi_x(x - \varepsilon s, t)}{\Psi(x - \varepsilon s, t)} ds \int_{-\infty}^{\infty} \Lambda'''(s) \ln \Psi(x - \varepsilon s, t) ds - \\ &\quad - \frac{2}{\varepsilon} \int_{-\infty}^{\infty} \Lambda'''(s) \frac{\Psi_x^2(x - \varepsilon s, t)}{\Psi^2(x - \varepsilon s, t)} ds. \end{aligned}$$

Let us estimate the integrals on the right-hand side. From (2.3)–(2.5) it follows that

$$\int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} \Lambda'''(s) ds = O(\varepsilon^2), \quad \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} s \Lambda'''(s) ds = -1 + O(\varepsilon^{3/2}).$$

Then we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \Lambda'''(s) \ln \Psi(x - \varepsilon s, t) ds &= \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} \Lambda'''(s) \left[\ln \Psi(x, t) - \varepsilon s \frac{\Psi_x(x, t)}{\Psi(x, t)} \right] ds + O(\varepsilon^2) = \\
&= \varepsilon \frac{\Psi_x(x, t)}{\Psi(x, t)} + O(\varepsilon^2), \\
\int_{-\infty}^{\infty} \Lambda'''(s) \frac{\Psi_x(x - \varepsilon s, t)}{\Psi(x - \varepsilon s, t)} ds &= -\varepsilon \left(\frac{\Psi_{xx}}{\Psi} - \frac{\Psi_x^2}{\Psi^2} \right) \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} s \Lambda'''(s) ds + O(\varepsilon^2) = \\
&= \varepsilon \left(\frac{\Psi_{xx}}{\Psi} - \frac{\Psi_x^2}{\Psi^2} \right) + O(\varepsilon^2), \\
\int_{-\infty}^{\infty} \Lambda'''(s) \frac{\Psi_x^2(x - \varepsilon s, t)}{\Psi^2(x - \varepsilon s, t)} ds &= -2\varepsilon \left(\frac{\Psi_x \Psi_{xx}}{\Psi^2} - \frac{\Psi_x^3}{\Psi^3} \right) \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} s \Lambda'''(s) ds + O(\varepsilon^2) = \\
&= 2\varepsilon \left(\frac{\Psi_x \Psi_{xx}}{\Psi^2} - \frac{\Psi_x^3}{\Psi^3} \right) + O(\varepsilon^2).
\end{aligned}$$

Consequently, we conclude that

$$R_t + RR_x - R_{xx} = O(\varepsilon).$$

Since $R(x, 0, \varepsilon) = \varepsilon \Lambda(x/\varepsilon)$, by the continuity of the inverse operator, we can also state that $R(x, t, \varepsilon)$ is an asymptotic approximation of the exact solution of the Cauchy problem (2.1)–(2.2).

3 Korteweg–de Vries equation

Studying properties of solutions of the Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \varepsilon \frac{\partial^3 u}{\partial x^3} = 0, \quad t \geq 0, \quad \varepsilon > 0, \quad (3.1)$$

is of indisputable interest for describing nonlinear wave phenomena [2]. Open questions concerning the behavior of solutions are still the subject of attention for modern researches [9]. Some mathematical results about the solution of the problem in various cases can be found in [10] and [11].

Consider the Cauchy problem for (3.1) with the initial condition

$$u(x, 0, \varepsilon, \rho) = \Lambda(x\rho^{-1}), \quad t = 0, \quad x \in \mathbb{R}, \quad \rho > 0. \quad (3.2)$$

Here, we assume that the initial function Λ is bounded and possesses the first derivative, which sufficiently fast tends to zero at infinity. It is required to find an asymptotic approximation of the solution $u(x, t, \varepsilon, \rho)$ to problem (3.1)–(3.2) in small parameters ε and ρ for

all x and finite values of t . It is clear that the structure of the asymptotics must depend on the relation between parameters ε and ρ . We assume the fulfillment of the condition

$$\mu = \frac{\rho}{\sqrt{\varepsilon}} \rightarrow 0.$$

In paper [12], a uniformly suitable asymptotic approximation of the solution to problem (3.1)–(3.2) is constructed using the renormalization method in the following most simple form. Let us pass to the inner variables

$$x = \sqrt{\varepsilon} \eta, \quad t = \sqrt{\varepsilon} \theta,$$

since this allows one to take into account all terms in equation (3.1). As a first approximation, we take the solution of the equation

$$\frac{\partial Z}{\partial \theta} + Z \frac{\partial Z}{\partial \eta} + \frac{\partial^3 Z}{\partial \eta^3} = 0, \quad (3.3)$$

with the initial condition

$$Z(\eta, 0) = \begin{cases} \Lambda_0^-, & \eta < 0, \\ \Lambda_0^+, & \eta > 0, \end{cases} \quad (3.4)$$

where $\Lambda_0^\pm = \lim_{s \rightarrow \pm\infty} \Lambda(s)$. As a model of collisionless shock waves, problem (3.3)–(3.4) was studied by A.V. Gurevich and L.P. Pitaevskii in [13].

Let us construct the expansion of the solution in the following form:

$$u(x, t, \varepsilon, \rho) = Z(\eta, \theta) + \mu W(\eta, \theta, \mu) + O(\mu^\alpha), \quad \alpha > 0, \quad (3.5)$$

where the addend $\mu W(\eta, \theta, \mu)$ must eliminate the singularity of Z at the initial moment of time. Then the function W satisfies the linear equation

$$\frac{\partial W}{\partial \theta} + \frac{\partial(ZW)}{\partial \eta} + \frac{\partial^3 W}{\partial \eta^3} = 0. \quad (3.6)$$

Differentiating equation (3.3) with respect to η , we find that the expression

$$G(\eta, \theta) = \frac{1}{\Lambda_0^+ - \Lambda_0^-} \frac{\partial Z(\eta, \theta)}{\partial \eta}$$

satisfies equation (3.6). Moreover, G is the Green function, because

$$\lim_{\theta \rightarrow +0} \int_{-\infty}^{\infty} G(\eta, \theta) f(\eta) d\eta = -\frac{1}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} Z(\eta, 0) f'(\eta) d\eta = f(0)$$

for any smooth function f with compact support, thus $G(\eta, 0) = \delta(\eta)$.

Let us choose the solution W in the form the convolution with the Green function G so that the asymptotic approximation would satisfy the initial condition (3.2). As a result, expansion (3.5) becomes

$$u(x, t, \varepsilon, \rho) = U_0(x, t, \varepsilon, \rho) + O(\mu^\alpha),$$

where

$$U_0(x, t, \varepsilon, \rho) = Z(\eta, \theta) + \frac{\mu}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} \frac{\partial Z(\eta - \mu s, \theta)}{\partial \eta} [\Lambda(s) - Z(s, 0)] ds.$$

Integrating by parts, we obtain the asymptotic approximation

$$u(x, t, \varepsilon, \rho) \approx U_0(x, t, \varepsilon, \rho) = \frac{1}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} Z\left(\frac{x - \rho s}{\sqrt{\varepsilon}}, \frac{t}{\sqrt{\varepsilon}}\right) \Lambda'(s) ds.$$

A rigorous mathematical justification of this formal representation is out of the framework of the present paper. It is clear that the behavior of the convolution integral is essentially determined by the solution of (3.3)–(3.4) with discontinuous initial data that confirms the importance of the Gurevich–Pitaevskii problem.

Constructing complete asymptotic expansions of the solution near the singular point by the standard matching method may be connected with serious difficulties. In fact, it is necessary to solve the scattering problem for a recurrence system of partial differential equations with variable coefficients [14]. In addition, the investigation of the shock wave generated by gradient catastrophe shows that the asymptotics of the solution in a neighborhood of a singular point may have a multiscale structure [8].

The renormalization approach allows one to construct a uniformly suitable asymptotics in the whole domain of independent variables avoiding difficulties arising from the matching procedure.

In particular, for $\Lambda_0^+ = 0$ and $\Lambda_0^- = a > 0$ the following formula was obtained in paper [12]:

$$u(x, t, \varepsilon, \rho) \approx 2\Lambda\left(\frac{x + at}{\rho}\right) - \Lambda\left(\frac{x - 2at/3}{\rho}\right) - \frac{at}{\rho} \int_{-1}^{2/3} \Lambda'\left(\frac{x - aty}{\rho}\right) \left[2 \operatorname{dn}^2\left(\frac{a^{3/2}t\omega(y)}{\sqrt{\varepsilon}}, \sigma(y)\right) + \sigma^2(y)\right] dy,$$

where $\operatorname{dn}(v, \sigma)$ is the elliptic Jacobi function

$$\operatorname{dn}(u, m) = \sqrt{1 - m \sin^2 \varphi}, \quad u = \int_0^{\varphi(u)} \frac{dv}{\sqrt{1 - m \sin^2 v}},$$

$$\omega(y) = \frac{1}{\sqrt{6}} \left\{ y - \frac{1}{3} [1 + \sigma^2(y)] \right\}, \quad 1 + \sigma^2 - \frac{2\sigma^2(1 - \sigma^2)K(\sigma)}{E(\sigma) - (1 - \sigma^2)K(\sigma)} = 3y,$$

$K(\sigma)$ and $E(\sigma)$ are complete elliptic integrals of first and second kind:

$$K(\sigma) = \int_0^{\pi/2} \frac{dv}{\sqrt{1 - \sigma^2 \sin^2 v}}, \quad E(\sigma) = \int_0^{\pi/2} \sqrt{1 - \sigma^2 \sin^2 v} dv.$$

Note that using the change $x = \rho\xi$, $t = \rho\tau$ for (3.1)–(3.2), we obtain the problem

$$\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial \xi} + \mu^{-2} \frac{\partial^3 u}{\partial \xi^3} = 0, \quad u|_{\tau=0} = \Lambda(\xi),$$

which formally contains only one parameter. However, in this case one should study asymptotics as $\mu^{-2} \rightarrow \infty$ and $\tau \rightarrow \infty$.

4 Perturbation of a weak discontinuity

Now consider the Cauchy problem for the KdV equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad (4.1)$$

with the initial condition

$$u(x, 0, \varepsilon) = \varepsilon \Lambda\left(\frac{x}{\varepsilon}\right), \quad t = 0, \quad x \in \mathbb{R}, \quad (4.2)$$

where $\Lambda(s) = -s \Theta(-s) + O(s^{-2})$ as $s \rightarrow \infty$, Θ is the Heaviside function. By analogy with the case of the Burgers equation we postulate the expression

$$R(x, t, \varepsilon) = \int_{-\infty}^{\infty} \frac{d^2 \Lambda(s)}{ds^2} \Phi(x - \varepsilon s, t) ds \quad (4.3)$$

as an asymptotic solution for $0 < t \leq \delta < 1$, where $\Phi(x, t)$ is a smooth solution of the Faminskii problem [15]

$$\Phi_t + \Phi \Phi_x + \Phi_{xxx} = 0, \quad \Phi(x, 0) = -x \Theta(-x) \quad (4.4)$$

with an initial weak discontinuity also called a contact discontinuity. It is easy to show that the function (4.3) exactly satisfies the initial condition (4.2).

Theorem 2. *Let $\Lambda(s)$ be a twice differentiable function and let*

$$\Lambda(s) = -s \Theta(-s) + O(s^{-2}), \quad \frac{d\Lambda(s)}{ds} = -\Theta(-s) + O(|s|^{-3}), \quad s \rightarrow \infty. \quad (4.5)$$

Then the function (4.3) is an asymptotic solution of the Korteweg–de Vries equation (4.1).

Proof. From (4.5) it follows that

$$\int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} \Lambda''(s) ds = 1 + O(\varepsilon^{3/2}), \quad \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} s \Lambda''(s) ds = O(\varepsilon).$$

Taking into account these relations, using (4.4), and substituting $u = R(x, t, \varepsilon)$ into (4.1), we find

$$\begin{aligned} R_t + R R_x + R_{xxx} &= \int_{-\infty}^{\infty} \Lambda''(s) \Phi(x - \varepsilon s, t) ds \int_{-\infty}^{\infty} \Lambda''(s) \Phi_x(x - \varepsilon s, t) ds - \\ &- \int_{-\infty}^{\infty} \Lambda''(s) \Phi(x - \varepsilon s, t) \Phi_x(x - \varepsilon s, t) ds = O(\varepsilon^{3/2}). \end{aligned}$$

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